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On some anisotropy-based analysis problems for linear discrete-time descriptor systems with nonzero-mean input signals

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Андрианова О. Г.

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Россия, МГТУ им. Н.Э. Баумана,
Россия, Институт проблем управления РАН
andrianovaog@gmail.com

В работе представлен новый подход к анизотропийному анализу дискретных дескрипторных систем в случае входных сигналов с ненулевым средним.

Дескрипторные системы являются обобщенным случаем обыкновенных систем. Они содержат как дифференциальные (разностные) уравнения, так и алгебраические. Когда переменные состояния имеют физический смысл, модели систем обычно представляют в дескрипторной форме.

В классической стохастической теории анизотропийного робастного управления рассматривают входные сигналы с нулевым математическим ожиданием и заданной «цветностью». В реальных технических системах на вход могут подаваться и стохастические сигналы с ненулевым средним. Именно поэтому распространение теории анизотропийного робастного управления на класс сигналов с ненулевым математическим ожиданием имеет практический интерес. Основными понятиями данной теории являются: анизотропия случайного вектора, средняя анизотропия входной последовательности и анизотропийная норма системы. Анизотропия случайного вектора характеризует «цветность» сигнала как меру отличия плотности распределения вероятности (п.р.в.) сигнала от п.р.в. гауссовского белого шума. Средняя анизотропия последовательности — это анизотропия, усредненная по времени. Анизотропийная норма представляет собой стохастический коэффициент усиления системы, когда на вход подается последовательность с заданным уровнем средней анизотропии.

В данной работе решена задача вычисления средней анизотропии стационарной гауссовской последовательности с ненулевым средним в случае, когда формирующий фильтр представлен в дескрипторной форме. Основываясь на полученном алгоритме, среднюю анизотропию последовательности можно вычислить в пространстве состояний, используя решения уравнений Риккати и Ляпунова, при этом формирующий фильтр записан во второй эквивалентной форме (SVD).

Для заданного уровня средней анизотропии входного сигнала получены уравнения вычисления анизотропийной нормы в частотной области (для дескрипторных систем). Приведен численный пример, который иллюстрирует технику вычисления анизотропийной нормы. Показано, что функции для вычисления анизотропийной нормы теряют монотонность, когда на вход подается сигнал с ненулевым средним. В связи с этим вычисление анизотропийной нормы в пространстве состояний остается сложной и нерешенной задачей.

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On some anisotropy-based analysis problems for linear discrete-time descriptor systems with nonzero-mean input signals

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Andrianova O. G.

Bauman Moscow State Technical University,
105005, Moscow, Russian Federation;
Institute of Control Sciences of Russian Academy of Sciences,
117997, Moscow, Russian Federation
andrianovaog@gmail.com

Introduction

The theory of optimal stochastic anisotropy-based control for linear discrete-time systems was established in Russia since 1994 [1], [2]. This theory allows to design control that minimizes specified norm of the closed-loop system (anisotropic norm). Since 2008 the theory of suboptimal stochastic anisotropy-based control that provides boundedness of anisotropic norm for the referred above systems was developed [3]. At present these mathematical theories find application in different control and filtration problems.

The created theory lies between the classical H_2-optimal and H_∞-optimal control theories (and suboptimal as well) in some sense. The basic concepts of these theories are anisotropy of the random signal and mean anisotropy of the input sequence. Anisotropy of the vector describes so-called "spectral color" of the signal as the distinction between its probability density function (p.d.f.) and p.d.f. of the Gaussian white noise sequence. Mean anisotropy of the sequence is the time averaging anisotropy.

In classical anisotropy-based theory the input signal is supposed to be the signal with zero mean and certain "spectral color". But in real technical systems the input signal can be a stochastic signal with nonzero mean. That is why the extension of anisotropy-based theory on the class of signals with nonzero mean has a practical interest. The formulas of anisotropy of the random vector, mean anisotropy of the signal and anisotropic norm of the system with a random nonzero-mean input signal were obtained in [4].

In this paper, anisotropy-based theory with zero mean input signals was generalized on linear discrete-time descriptor systems. Mathematical models of such systems contain algebraic equations. Algebraic equations in the model of the system appear as constraints when the system variables have the meaning of physical processes. Descriptor systems (or singular systems, semistate systems, degenerate systems) are a general case of normal systems, described by ordinary differential or difference equations. Because of the algebraic relations between state variables, the system becomes singular and has specific behavior, different from normal systems, for example, impulsive behavior in continuous time case or noncausal behavior in discrete time case. So it is necessary to generalize mathematical methods, developed for normal systems, on descriptor case.

Optimal control problem was solved in [5], [6], norm boundedness conditions were obtained in [7], [8]. This paper presents the extension of anisotropy-based theory on the class of descriptor systems in assumption that the "colored" Gaussian noise has nonzero mean.

The paper is organized as follows. In the first section, the basics of linear discrete-time descriptor systems is introduced. The second section deals with anisotropy-based concepts, applied to the Gaussian stationary random sequences with nonzero mean. The third section is devoted to anisotropy-based analysis of descriptor systems in frequency domain. Numerical examples are given.

1. Basics of descriptor systems theory

In linear case discrete-time descriptor systems are written as

$$\begin{cases} Ex_{k+1} = Ax_k + Bu_k, \\ y_k = Cx_k + Du_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^{n_1}$ is the state, $u_k \in \mathbb{R}^m$ is the control signal and $y_k \in \mathbb{R}^p$ is the output signal. A, B, C, D are constant real matrices of appropriate dimensions.

For the system (1) we suppose that $\operatorname{rank}(E) = n < n_1$. Such systems are called descriptor or singular.

Some basic properties of descriptor systems are associated with matrices E and A. In different literature [9, 10], they can be represented as a matrix pencil $(zE - A)$ or a pair (E, A).

Definition 1. The pair (E, A) is said to be regular if there exists a scalar λ such that $\det(\lambda E - A) \neq 0$.

Regularity of the pair (E, A) is a necessary and sufficient condition of existence and uniqueness of solution of the system (1). The following lemma [9] provides necessary and sufficient conditions of regularity for the system (1).

Lemma 1. The pair (E, A) is regular if and only if there exist invertible matrices Q_1 and U_1 such that

$$\bar{E} = Q_1 E U_1 = \operatorname{diag}(I_n, N), \quad \bar{A} = Q_1 A U_1 = \operatorname{diag}(A_1, I_{n_1 - n}) \tag{2}$$

where $A_1 \in \mathbb{R}^{n \times n}$, N is a nilpotent matrix, that is, $N^h = 0$ for some positive integer h. The minimum h_0 such that $N_0^h = 0$ is called the index of N.

The index of the system (1) in equivalent form (2) is called the index of the nilpotent N. $(\bar{E}, \bar{A})$ is called the Weierstrass canonical form of (E, A).

Definition 2. The system (1) is called causal if its solution x_k depends only on $u_k, \ldots, u_0$ and $x_{k-1}, \ldots, x_0$ for any consistent initial conditions. It is true if the index of the nilpotent N is equal to 1.

The system (1) is causal if $\deg \det(zE - A) = \operatorname{rank} E$.

Definition 3. The system (1) is called stable if $\rho(E, A) < 1$ where

$$\rho(E, A) \triangleq \max |\lambda|_{\lambda \in \{z |\, \det(zE - A) = 0\}}$$

is a generalized spectral radius of the pair (E, A).

Definition 4. The system (1) is said to be admissible if the pair (E, A) is regular, and the system (1) is stable and causal. For more information, see [9].

It is known from matrix theory, that there are two nonsingular matrices Q and U such that $QEU = \operatorname{diag}(I_n, 0)$. By the following transformation of coordinates $U^{-1}x_k = \begin{pmatrix} x_{1,k} \\ x_{2,k} \end{pmatrix}$, $x_{1,k} \in \mathbb{R}^n$, $x_{2,k} \in \mathbb{R}^{n_1 - n}$ the system (1) can be written in the following equivalent form:

$$\begin{cases} x_{1,k+1} = A_{11}x_{1,k} + A_{11}x_{2,k} + B_1 u_k, \\ 0 = A_{21}x_{1,k} + A_{22}x_{2,k} + B_2 u_k, \\ y_k = C_1 x_{1,k} + C_2 x_{2,k} + D u_k \end{cases} \tag{3}$$

where matrices A_{ij}, B_i, C_i ($i,\ j = 1,\ 2$) satisfy

$$QAU = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad QB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad CU = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

Matrices Q and U can be found from the singular value decomposition (SVD) [5].

Definition 5. The system (3) is called SVD canonical form of the system (1).

For the system (1) in SVD canonical form (3) the following lemma [10] holds true.

Lemma 2. The system (1) is

1) causal if and only if A_{22} is nonsingular;

2) admissible if and only if it is casual and $\rho(A_{11} - A_{12}A_{22}^{-1}A_{21}) < 1$.

Definition 6. The transfer function of the system (1) is given by the expression $P(z) = C(zE - A)^{-1}B + D$.

Definition 7. Let $L_2^{p\times m}(\Gamma)$ (where Γ is a unit circle on the complex plane) be the Hilbert space of matrix-valued functions $P : \Gamma \to C^{p\times m}$ that have bounded $L_2^{p\times m}(\Gamma)$-norm

$$\|P\|_{L_2^{p\times m}(\Gamma)} = \left(\frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}\big(P^*(e^{i\omega})P(e^{i\omega})\big)\, d\omega \right)^{1/2}.$$

A subspace of $L_2^{p\times m}(\Gamma)$ which consists of all rational transfer functions that have no poles in the exterior of the closed unit disk is denoted by H_2. The H_2-norm of a transfer function $P(z) \in H_2$ is defined by

$$\|P\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\omega})|^2 d\omega \right)^{1/2}.$$

2. Anisotropy of the random vector, mean anisotropy of the Gaussian sequence with nonzero mean

In this section, we consider the concept of anisotropy of the random vector with nonzero mean and give definition of mean anisotropy of the Gaussian nonzero-mean sequence, generated by the shaping filter in descriptor form.

Anisotropy of the random vector. Anisotropy of m-dimensional random vector w is introduced in [1] as the minimal value of the relative entropy of w with respect to the Gaussian m-dimensional vector with p.d.f.

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2} \exp\left(-\frac{x^T x}{2\lambda}\right), \quad x \in \mathbb{R}^m,$$

and is described by

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{E}_f \ln \frac{f(x)}{p_{m,\lambda}(x)} \tag{4}$$

where the function f is p.d.f. of w.

We suppose w is m-dimensional Gaussian random vector with nonzero mean μ and covariance matrix S, which p.d.f. is given by

$$f(x) = ((2\pi)^m |S|)^{-1/2} \exp\left(-\frac{(x-\mu)^T S^{-1}(x-\mu)}{2}\right), \quad x \in \mathbb{R}^m.$$

By definition of anisotropy (4) of the random vector,

$$\mathbf{A}(w) = -\frac{1}{2} \ln \det\left(\frac{mS}{\mathrm{tr}S + |\mu|^2}\right).$$

One can show that if $S = \gamma I_m$ and $\mu = 0$, then $\mathbf{A}(w) = 0$ where γ is a constant real value.

Mean anisotropy of the Gaussian sequence with nonzero mean. Let W be a stationary sequence of random m-dimensional vectors. Mean anisotropy of the stationary ergodic sequence $W = \{w_k\}$ is defined in [1] by the following expression

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N}$$

where $W_{0:N-1}$ is an extended vector of the sequence:

$$W_{0:N-1} = \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix}.$$

Let the sequence $W = \{w_k\}$ be generated from the Gaussian white noise $V = \{v_k\}$ by an admissible shaping filter

$$G \sim \begin{cases} E_g x_{k+1} = A_g x_k + B_g(v_k + \mu), \\ w_k = C_g x_k + D_g(v_k + \mu) \end{cases} \tag{5}$$

where $E_g \in \mathbb{R}^{n_1\times n_1}$, $A_g \in \mathbb{R}^{n_1\times n_1}$, $B_g \in \mathbb{R}^{n_1\times m}$, $C_g \in \mathbb{R}^{m\times n_1}$, $D_g \in \mathbb{R}^{m\times m}$.

Besides, $\det(D_g) \neq 0$, $\operatorname{rank} E_g = n < n_1$ and $|\mu| < \infty$. Connection between mean anisotropy $\overline{\mathbf{A}}(W)$ of the sequence W and state-space representation (5) of shaping filter is given by the following theorem.

Theorem 1. For a given state-space representation (5) of the shaping filter G mean anisotropy $\overline{\mathbf{A}}(W)$ is determined by

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\det\left(\frac{m(\Sigma+\Xi)}{\mathrm{tr}\Sigma + |\mathcal{M}|^2}\right)$$

where Σ and Ξ are connected with the solutions of Lyapunov and Riccati equations P and R by formulas

$$\Sigma = \widehat{C}P\widehat{C}^T + \widehat{D}\widehat{D}^T, \quad P = \widehat{A}P\widehat{A}^T + \widehat{B}\widehat{B}^T, \quad \Xi = \widehat{C}R\widehat{C}^T,$$
$$R = \widehat{A}R\widehat{A}^T - \Lambda(\Sigma+\Xi)^{-1}\Lambda^T, \quad \Lambda = \widehat{B}\widehat{D}^T + \widehat{A}(P+R)\widehat{C}^T$$

with matrices

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$
$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \widehat{D} = D_g - C_2A_{22}^{-1}B_2,$$

connected with matrices A_{ij}, B_i, C_i $(i,j = 1,2)$ of SVD canonical form of the system (5), and the vector $\mathcal{M}$ is represented by

$$\mathcal{M} = (\widehat{D} + \widehat{C}(I_{n\times n} - \widehat{A})^{-1}\widehat{B})\mu.$$

Д о к а з а т е л ь с т в о. The system (5) in SVD canonical form can be rewritten as

$$\begin{cases} x^1_{k+1} = A_{11}x^1_k + A_{12}x^2_k + B_1(v_k+\mu), \\ 0 = A_{21}x^1_k + A_{22}x^2_k + B_2(v_k+\mu), \\ w_k = C_1x^1_k + C_2x^2_k + D_d(v_k+\mu) \end{cases} \tag{6}$$

where $x^1_k \in \mathbb{R}^n$, $x^2_k \in \mathbb{R}^{n_1-n}$. According to Lemma 1, $\det A_{22} \neq 0$, then

$$x^2_k = -A_{22}^{-1}(A_{21}x^1_k + B_2(v_k+\mu)). \tag{7}$$

Substituting x^2_k into the first and the third subsystems of (6), one can get

$$\begin{cases} x^1_{k+1} = \widehat{A}x^1_k + \widehat{B}(v_k+\mu), \\ w_k = \widehat{C}x^1_k + \widehat{D}(v_k+\mu) \end{cases} \tag{8}$$

where

$$\widehat{A}=A_{11}-A_{12}A_{22}^{-1}A_{21}, \quad \widehat{B}=B_1-A_{12}A_{22}^{-1}B_2, \quad \widehat{C}=C_1-C_2A_{22}^{-1}A_{21}, \quad \widehat{D}=D_g-C_2A_{22}^{-1}B_2.$$

Applying Theorem 1 from [4] to the reduced normal system, we can finish the proof.

The random sequence W is fully defined by its generating filter G, therefore, the notation $A(G)$ is used as equivalent to the notation $A(W)$.

Example 1. Let the shaping filter (5) be formed by the following matrices

$$E_g = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 1 & 1 & 2 \end{pmatrix}, \quad B_g = \begin{pmatrix} 0.03 \\ 0.10 \\ 0.07 \end{pmatrix}, \quad A_g = \begin{pmatrix} 0.7649 & 0.7572 & -0.0581 \\ -0.0424 & 0.2854 & 0.2218 \\ 0.7706 & 0.6003 & 0.7157 \end{pmatrix},$$

$$C_g = (1 \quad 2 \quad 1.5), \quad D_g = (0.5),$$

and vector $\mu = (0.1)$; $\operatorname{rank} E = 2$, $m = 1$. The system in SVD canonical form is defined by matrices:

$$\widehat{A} = \begin{pmatrix} 0.7187 & 0.0253 \\ 0.9639 & -0.3064 \end{pmatrix}, \quad \widehat{B} = \begin{pmatrix} -0.1213 \\ -0.2673 \end{pmatrix}, \quad \widehat{C} = (-2.2291 \quad 0.9010), \quad \widehat{D} = (0.7324).$$

So, the vector $\mathcal{M} = (0.065)$. Solving Lyapunov and Riccati equations from the Theorem 1, we obtain

$$\Sigma = (0.5873), \quad \Xi = (-0.0510).$$

Consequently,

$$\overline{\mathbf{A}}(W) = -\frac{1}{2} \ln \det \left(\frac{m(\Sigma + \Xi)}{\operatorname{tr}\Sigma + |\mathcal{M}|^2} \right) = 0.049.$$

3. Anisotropic norm

Consider an admissible linear discrete-time descriptor system written in a state-space representation

$$P \sim \begin{cases} Ex_{k+1} = Ax_k + Bw_k, \\ z_k = Cx_k + Dw_k \end{cases} \tag{9}$$

where $x_k \in \mathbb{R}^{n_1}$ is the state, $w_k \in \mathbb{R}^m$ and $z_k \in \mathbb{R}^p$ are input and output signals, respectively. E, A, B, C, D are constant real matrices of appropriate dimensions. We suppose that the matrix E is singular, i.e. $\operatorname{rank}(E) = n < n_1$. $W = \{w_k\}$ is the stationary Gaussian sequence of m-dimensional random vectors with a given mean anisotropy level $\overline{\mathbf{A}}(W) = a \geq 0$ and known nonzero mean $\mathbf{E}w_\infty = \mathcal{M}$, $|\mathcal{M}| < \infty$.

For a given system P with the input signal $W = \{w_k\}$ the mean-square gain is defined as [12]

$$Q(P, W) = \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}} = \sqrt{\lim_{N\to\infty} \frac{\frac{1}{N}\sum_{k=0}^{N-1} \mathbf{E}|z_k|^2}{\frac{1}{N}\sum_{k=0}^{N-1} \mathbf{E}|w_k|^2}} \tag{10}$$

where

$$\|y\|_{\mathcal{P}} = \sqrt{\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{E}|y_k|^2}$$

is the power norm of the signal $\{y_k\}$.

Let the sequence $\{w_k\}$ be represented in the form

$$w_k = C_g x_k + D_g(v_k + \mu) \tag{11}$$

where x_k is the state of the system (9), and μ is a known vector. Using (11), we obtain the admissible filter

$$G \sim \begin{cases} E x_{k+1} = (A + BC_g)x_k + BD_g(v_k + \mu), \\ w_k = C_g x_k + D_g(v_k + \mu). \end{cases} \tag{12}$$

The power norms of outputs of the systems (9) and (12) are written as

$$\|w\|_{\mathcal{P}}^2 = \lim_{k\to\infty} \left(\mathrm{tr}\ \mathrm{cov}(w_k) + |\mathbf{E}w_k|^2\right) = \|G\|_2^2 + |\mathcal{M}|^2,$$

$$\|z\|_{\mathcal{P}}^2 = \lim_{k\to\infty} \left(\mathrm{tr}\ \mathrm{cov}(z_k) + |\mathbf{E}z_k|^2\right) = \|PG\|_2^2 + |\mathcal{P}\mathcal{M}|^2$$

where

$$\mathcal{P} = P(1) = D + C(E - A)^{-1}B.$$

The mean-square gain (10) of the system with nonzero-mean input signal is given by the following expression:

$$Q(P, W) = Q(P, G) = \sqrt{\frac{\|PG\|_2^2 + |\mathcal{P}\mathcal{M}|^2}{\|G\|_2^2 + |\mathcal{M}|^2}}. \tag{13}$$

Finally, anisotropic norm of the system is defined as [2]

$$|\!|\!|P|\!|\!|_a = \sup_{G:\overline{\mathbf{A}}(G)\le a} Q(P, G). \tag{14}$$

Theorem 2. Consider the system, defined by (9) (with the transfer function $P(z) = C(zE - A)^{-1}B + D$). Let W be the sequence of nonzero-mean m-dimensional Gaussian random vectors, generated by an admissible shaping filter G in the form (5), with mean anisotropy $\overline{\mathbf{A}}(W) = a$ and $\mathbf{E}w_\infty = \mathcal{M}$. Then anisotropic norm of the descriptor system (9) can be computed in the frequency domain as

$$|\!|\!|P|\!|\!|_a = \sup_{q\in[0;\|P\|_\infty^{-2})} \{N(q) \mid A(q) = a\} \tag{15}$$

where

$$A(q) = \frac{m}{2}\left(\ln\left(\Phi(q) + \frac{1}{m}|\mathcal{M}|^2\right) - \Psi(q)\right), \quad N(q) = \sqrt{\frac{\Phi(q) - 1 + \frac{q}{m}|\mathcal{P}\mathcal{M}|^2}{q\Phi(q) + \frac{q}{m}|\mathcal{M}|^2}},$$

$$\Phi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \mathrm{tr}S(q,\omega)d\omega, \quad \Psi(q) = \frac{1}{2\pi m}\int_{-\pi}^{\pi} \ln\det S(q,\omega)d\omega,$$

$$S(q,\omega) = (I_m - q\Lambda(\omega))^{-1}, \quad q \in [0; \|F\|_\infty^2).$$

Here $\Lambda(\omega) = \widehat{P}^*(\omega)\widehat{P}(\omega)$, $\widehat{P}(\omega) = \lim_{r\to 1} P(re^{j\omega})$, and $\mathcal{P} = P(1)$.

Besides,

$$N(0) = \sqrt{\frac{\|P\|_2^2 + |\mathcal{P}\mathcal{M}|^2}{m + |\mathcal{M}|^2}}.$$

The proof can be found in [11]. This theorem presents the equations of anisotropic norm computation (in the frequency domain) for descriptor systems. The functions $A(q)$ and $N(q)$ for anisotropic norm computation lose monotony when the input signal has nonzero mean. The performance of these functions in such case is shown and analyzed in the following example.

Example 2. Let the system P be described by

$$E = \begin{pmatrix} 0.9 & 0 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0.7 & -0.3 \\ 0.1 & 0.3 \end{pmatrix}, \quad B = \begin{pmatrix} -0.02 \\ 0.07 \end{pmatrix}, \quad C = \begin{pmatrix} 0.50 & 0.09 \end{pmatrix}, \quad D = \begin{pmatrix} 0.035 \end{pmatrix}.$$

The transfer function of the system is

$$P(z) = \frac{0.235}{9z - 8} + 0.014.$$

The spectral density of the system is

$$\Lambda(\omega) = \frac{0.031(1 + \cos\omega)}{-144\cos\omega + 145}.$$

The spectral density of the worst case shaping filter is

$$S(q, \omega) = \frac{144\cos\omega + 145}{(-144 - 0.031q)\cos\omega + 145 - 0.031q}.$$

Fig.1

fig.2

Fig. 1 and fig. 2 present $A(q)$ and $N(q)$ plots for different values of $\mathcal{M}$, respectively.

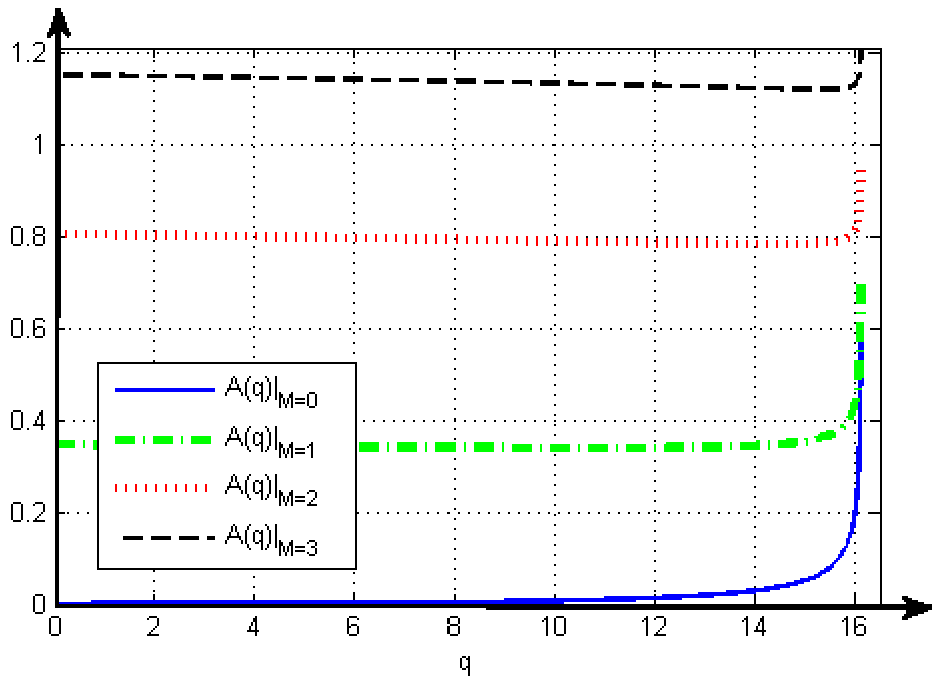

Рис. 1. $A(q)$ for different values of $\mathcal{M}$

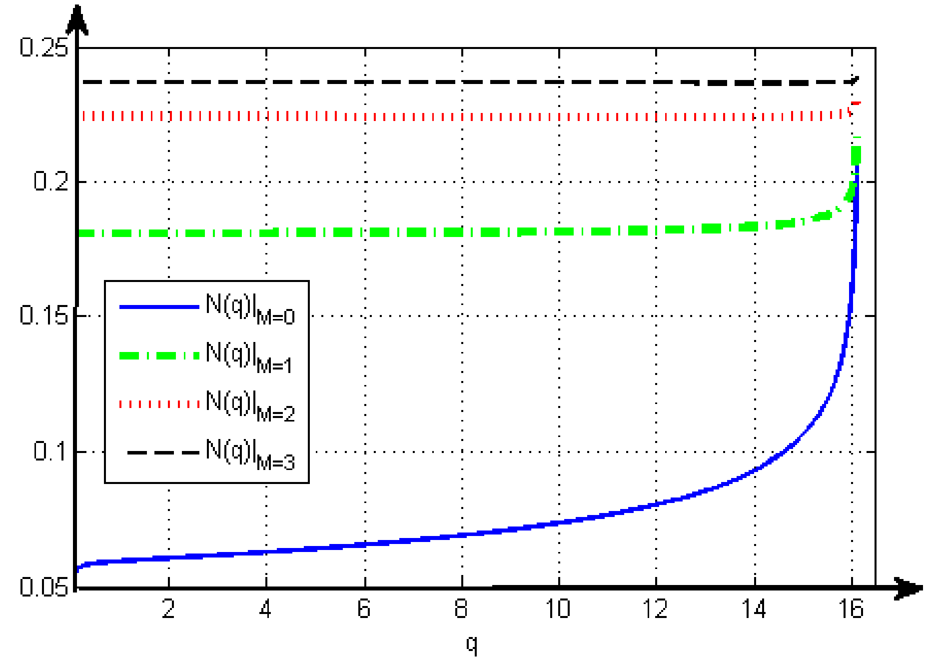

Рис. 2. $N(q)$ for different values of $\mathcal{M}$

fig.3

For large values of $\mathcal{M}$ functions $A(q)$ and $N(q)$ lose their monotony (see fig. 3). The set $\{N(q) \mid_{A(q)=a}\}$ can be empty or contain several values of $N(q)$. Therefore, the anisotropic norm is defined as a supremum function by (15).

fig.4

fig.5

When $a = 0.34$, the anisotropic norm of the system is equal to $|||P|||_a = 0.1841$ for $\mathcal{M} = 0$, $|||P|||_a = 0.1823$ for $\mathcal{M} = 1$ and cannot be computed for $\mathcal{M} = \{2, 3\}$ (see fig. 4 and 5).

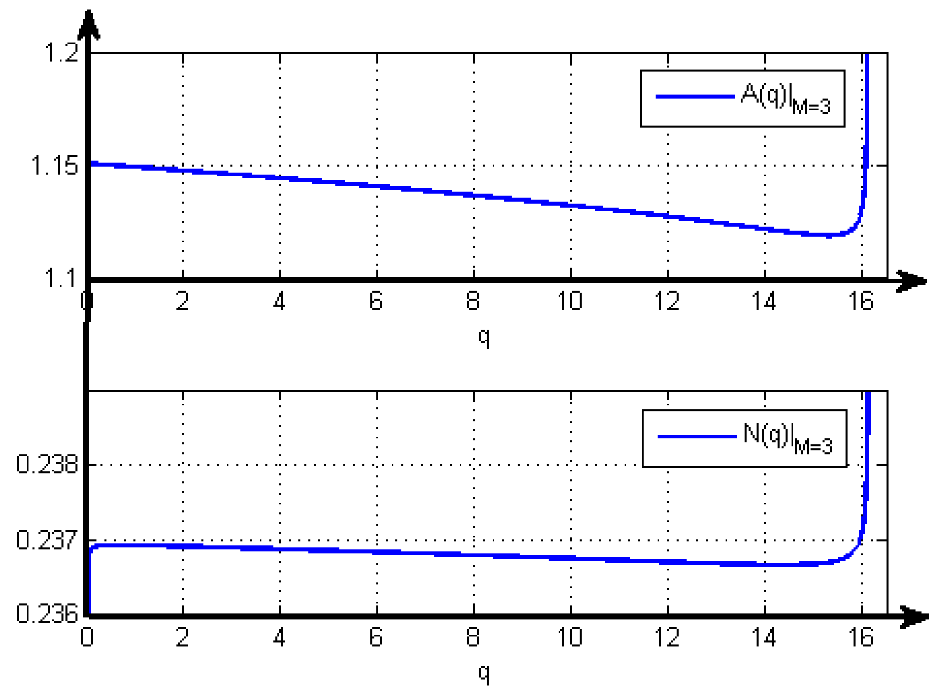

Рис. 3. $A(q)$ and $N(q)$ for $\mathcal{M} = 3$

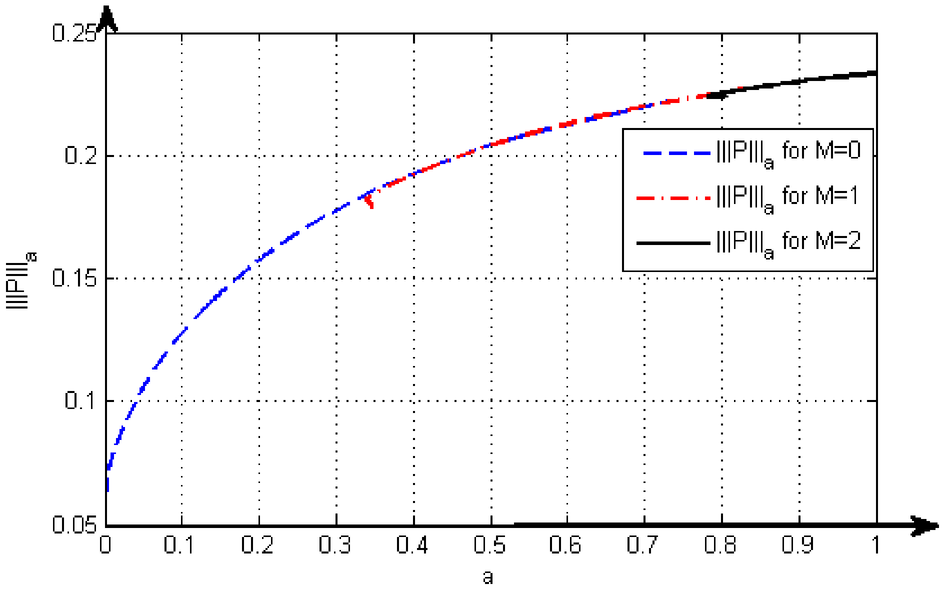

Рис. 4. $N(A(q))$ for different $\mathcal{M}$

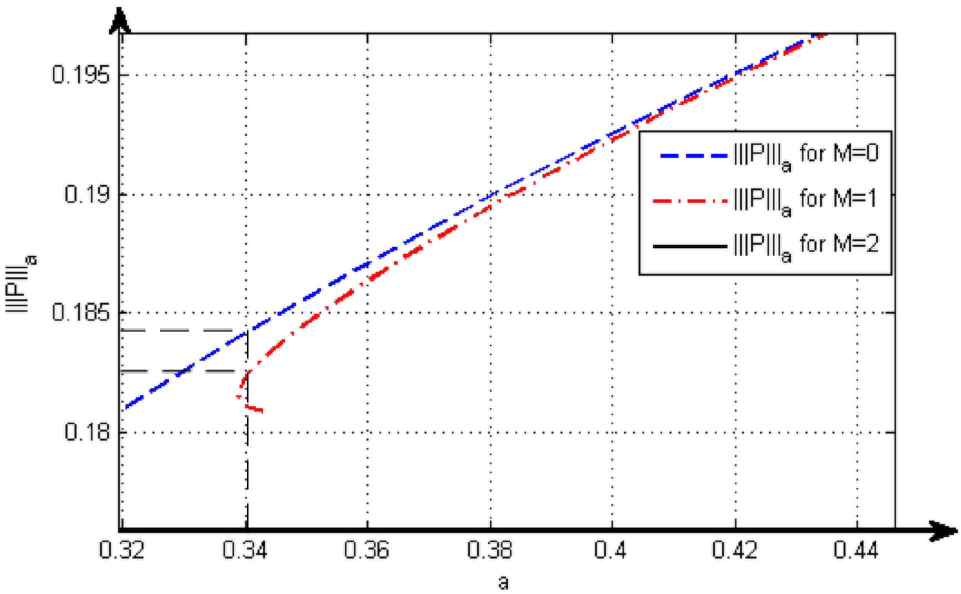

Рис. 5. $N(A(q))$ plots for different $\mathcal{M}$ (magnified)

Conclusions

In this paper, the problem of anisotropy-based analysis for linear discrete-time descriptor systems with Gaussian stationary random nonzero-mean sequences as input signals is solved. The algorithm of mean anisotropy computation of the Gaussian stationary random sequence with nonzero mean, generated by the shaping filter in descriptor form, is obtained. On basis of this algorithm the mean anisotropy of the sequence may be calculated in the state-space representation using the solutions of Lyapunov and Riccati equations. For the given mean anisotropy level of the input signal the equations of anisotropic norm computation (in the frequency domain) for descriptor systems are developed. Numerical example illustrates that for the given system functions for anisotropic norm computation lose monotony when the input signal has nonzero mean. This fact makes it difficult to compute anisotropic norm in the state-space representation and requires further investigation.

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